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GREEN'S FUNCTION OF ASYMMETRIC CHARACTERISTIC INITIAL  
BOUNDARY VALUE PROBLEMS FOR A HYPERBOLIC EQUATION

ABSTRACT

of the PhD thesis for the degree of  
doctor of Philosophy (PhD) in the specialty  
«6D060100-Mathematics»

**The relevance of the research topic.** In this PhD thesis, we investigate the problem of constructing the Green's function for boundary value problems for a hyperbolic equation. The Green's function for the hyperbolic problems differs significantly from the Green's function of problems for equations of elliptic and parabolic types.

For the problems for the equations of elliptic and parabolic types it can show that the difference of two Green's functions

$$g(x, y) = G_Q(x, y) - G_P(x, y)$$

is already a "smoother" function and is a solution of the homogeneous equation

$$L g(x, y) = 0, \quad x, y \in \Omega.$$

However, there is no longer the case for the problems for the equations of hyperbolic type. The function  $g(x, y)$  can also have singularities of the same order as the Green's functions  $G_Q(x, y)$  and  $G_P(x, y)$ .

Thus, if for the problems for the equations of elliptic and parabolic types the Green's function can be represented in the form of the sum of the "main part with singularities" and the "smooth term"

$$G_P(x, y) = G_Q(x, y) - g(x, y),$$

then there is no longer the case for the hyperbolic boundary value problems. This fact significantly complicates the consideration and therefore, for each separate case of boundary value problems, a separate investigation is required.

**The aim of the PhD thesis** is to substantiate the Green's function method for asymmetric characteristic initial-boundary value problems for the hyperbolic equation in a characteristic triangle.

**To achieve the aim of the dissertation, the main objectives** of the following research are considered:

-Construction of the Green function for the first initial boundary value problem in the quarter plane for a general two-dimensional hyperbolic equation of the second order;

-Construction of the Green function for the second initial boundary value problem in the quarter plane for a general two-dimensional hyperbolic equation of the second order;

-Construction of the Green function of the Darboux problem for a general hyperbolic equation in a characteristic triangle with a boundary condition of the first kind on an uncharacteristic boundary;

-Construction of the Green function for asymmetric characteristic boundary value problem for a general hyperbolic equation in a characteristic triangle with a boundary condition of the first kind on a non-characteristic boundary;

-Construction of the Green function for asymmetric characteristic boundary value problems for a general hyperbolic equation in a characteristic triangle with a boundary condition of the second kind on a non-characteristic boundary;

-Construction of an example of a correct characteristic boundary value problem having a "non-classical" form of the Green function;

-Construction of boundary conditions of a volume hyperbolic potential in a domain with a curvilinear boundary.

**Object of the PhD thesis** is a general second-order hyperbolic equation with variable coefficients.

**The methods of scientific research.** The dissertation uses methods of the theory of partial differential equations, the theory of functions, the theory of potential, the theory of special functions and the theory of differential geometry.

**Scientific novelty of the work.** The problems that considered in this dissertation are new.

One of the most significant achievements in the construction of the Green's function of a characteristic boundary value problem was the work of Haws L. Symmetric Green's functions for certain hyperbolic problems (1991). He considered a two-dimensional hyperbolic equation

$$u_{xy} + p(x, y)u = f(x, y), \quad (x, y) \in \Gamma,$$

in the characteristic triangle  $\Gamma = \{(x, y): 0 < x < y < 1\}$ .

One of the following two conditions was chosen as the boundary condition on the non-characteristic line  $AB$ : either a boundary condition of the first kind

$$u(x, x) = 0, \quad 0 \leq x \leq 1,$$

or a boundary condition of the second kind

$$(u_x - u_y)(x, x) = 0, \quad 0 \leq x \leq 1,$$

and the condition at point  $C$  was used:

$$u(0,1) = 0.$$

The problem was supplemented by another condition that ensures the symmetry of the Green's function of the problem under consideration. As an example, it was proposed to use self-adjoint boundary conditions with displacement.

In general, the problem was only formulated, but not solved. For the special case  $p(x, y) \equiv 0$ , the definition of the Green function was given and the method of its construction was specified.

The author also considered a hyperbolic equation with lower order coefficients

$$u_{xy} + au_x + bu_y + cu = f(x, y), \quad (x, y) \in \Gamma.$$

However, only the case of constant coefficients  $a, b, c$  was considered.

The problem of constructing the Green's function of a characteristic boundary value problem with arbitrary coefficients (without using the symmetry condition) was singled out as an unsolved problem, interesting for further consideration.

**Theoretical and practical significance of the results.** The research on the topic is mainly theoretical and fundamental. Their scientific significance is due precisely to the deep level of fundamentality of the results obtained.

**Publications.** According to the results of the dissertation, 10 papers were published: 3 journal articles (1 in journals indexed by Scopus, and 3 in a journal recommended by the Committee on the Control of Education and Science of the MES RK), 4 papers in materials of international scientific conferences (1 in Springer Proceedings).

The results on the topic of the thesis were published in the following papers:

#### Publication in the high-ranking scientific journals

1 M. A. Sadybekov, B. O. Derbissaly. On Green's function of Cauchy–Dirichlet problem for hyperbolic equation in a quarter plane // *Boundary Value Problems*. V. 69, 23 pp., 2021. Web of Science Impact factor=1,7(Q1), Scopus SJR=0,573 (Q1), CiteScore=3,5, Scopus Percentile=92

#### CCES

1 M. A. Sadybekov, B. O. Derbissaly. Boundary conditions of volume hyperbolic potential in a domain with curvilinear boundary // *Kazakh Mathematical Journal*. V. 19, № 4, P. 20-29, 2019.

2 M. A. Sadybekov, B. O. Derbissaly. On Green's function of Darboux problem for hyperbolic equation // Bulletin of KazNU. Series of mathematics, mechanics, computer science. V. 111, № 3, P. 79-94, 2021.

3 B. O. Derbissaly. On Green's function of the second Darboux problem for hyperbolic equation // Bulletin of KazNU. Series of mathematics, mechanics, computer science. V. 116, № 4, P. 3-14, 2022.

#### Kazakh local journal

1 M. A. Sadybekov, B. O. Derbissaly. On the Green function of the Cauchy-Neumann problem for the hyperbolic equation in the quarter plane // Kazakh Mathematical Journal. V. 21, № 1, P. 89-107, 2021.

#### Foreign journal

1 T. Sh. Kalmenov, B. O. Derbissaly. On the boundary conditions of the wave potential in a domain with curvilinear borders // Reports of Adyghe (Circassian) International Academy of Sciences. V 19, № 1, P. 22-31, 2019.

#### Publications in materials of international conferences

1 M. A. Sadybekov, B. O. Derbissaly. Boundary conditions of the hyperbolic potential in a domain with curvilinear boundary // Springer Proceedings in Mathematics & Statistics, V. 351, P. 257-271, 2021. Scopus SJR=0,253, Scopus Percentile=14

2 B. O. Derbissaly. On the Green function of the first initial-boundary problem of a hyperbolic equation in a quarter plane // Тезисы докладов: ежегодная научная апрельская конференция института математики и математического моделирования, посвященная 1150-летию Абу Насыр аль-Фараби и 75-летию Института математики и математического моделирования. С. 39, 2020.

3 Б. О. Дербисалы. Краевые условия объемного гиперболического потенциала в области с криволинейной границей // Тезисы докладов: ежегодная научная апрельская конференция института математики и математического моделирования и Workshop «Problems of modelling processes in electrical contacts», посвященный 80-летию юбилею академика НАН РК Станислава Николаевича Харина. С. 55, 2019.

4 M. A. Sadybekov, B. O. Derbissaly. On Green's function of asymmetric characteristic boundary value problem for hyperbolic equation in a characteristic triangle // International Conference "Numerical Functional Analysis", 22-24 ноября, 2021, Istanbul, Turkey

**The structure and scope of the thesis.** The PhD thesis consists of a title page, content, introduction, seven sections, conclusion and a list of references. The total volume of the dissertation is 108 pages with 68 references to literature.

**The main content of the thesis.** The introduction contains the relevance of the research topic, goals and objectives, the main provisions for the defense of the

dissertation, the object and subject of the research, research methods, novelty and theoretical and practical significance of the research, the connection of the PhD thesis with other research papers, the approbation of the work, the author's publications, the scope and structure of the dissertation and content.

In section 1, a definition of the Green function is given and a method for constructing of the Green function for the first initial boundary value problem in the quarter plane for a general two-dimensional hyperbolic equation of the second order is substantiated.

In section 2, a definition of the Green function is given and a method for constructing of the Green function for the second initial boundary value problem in the quarter plane for a general two-dimensional hyperbolic equation of the second order is substantiated.

In section 3, the definition of the Green function of the Darboux problem for a general hyperbolic equation considered in a characteristic triangle with a boundary condition of the first kind on an uncharacteristic boundary is given and the methodology for its construction is substantiated.

In section 4, the definition of the Green function for asymmetric characteristic boundary value problems for a general hyperbolic equation considered in a characteristic triangle with a boundary condition of the first kind on a non-characteristic boundary is given and the methodology for its construction is substantiated.

In section 5, the definition of the Green function for asymmetric characteristic boundary value problems for a general hyperbolic equation considered in a characteristic triangle with a boundary condition of the second kind on a non-characteristic boundary is given and the methodology for its construction is substantiated.

In Section 6, the examples of a correct characteristic boundary value problem having a "non-classical" form of the Green function are constructed.

In Section 7, the boundary conditions of a volume hyperbolic potential in a domain with a curvilinear boundary are constructed.

In conclusion, the main results obtained during the PhD thesis are presented.